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Yan Theorem in L^∞ with Applications to Asset Pricing

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Abstract We prove an L^∞ version of the Yan theorem and deduce from it a necessary condition for the absence of free lunches in a model of financial markets, in which asset prices are a continuous $\mathbb{R}^d$ valued process and only simple investment strategies are admissible. Our proof is based on a new separation theorem for convex sets of finitely additive measures.

Keywords Arbitrage, free lunch, fundamental theorem of asset pricing, martingale measure, Yan theorem
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1 Introduction

In a well known paper [23], Yan proved a result concerning convex subsets of L^1 which turned out to be influencial on much of the mathematical finance literature that followed. Given a probability P and a convex set $\mathcal{K} \subset L^1$ containing 0, the theorem provides a necessary and sufficient condition for the existence of a probability Q equivalent to P, and with respect to which the expected value of elements of $\mathcal{K}$ is uniformly bounded from above, i.e. $Q[\mathcal{K}] < a < \infty$. Several versions of this theorem have later been proved in the literature. Ansel and Stricker[1] obtained the first generalization to L^p, $1 \le p < \infty$; furthermore they illustrated the far reaching implications for mathematical finance, especially for arbitrage theory (see further development in this direction in [22]). Jouini, Napp and Schachermayer[11] have recently obtained a proof for locally convex spaces satisfying certain special conditions. In this paper we focus on L^∞ and show that the original claim of Yan remains true also in this context – see Theorem 2 below. We then apply this result to prove a version of the Fundamental Theorem of Asset Pricing (FTAP) for a continuous price process. Our argument relies on a preliminary result, stated in Theorem 1, concerning the separation of convex sets of finitely additive measures and from which the theorem of Yan readily follows. This general theorem may be of interest on its own.

In the literature there have been several different proofs of the *FTAP* (see, among others, [1,7,14] for the case of continuous prices process and [6] for the locally bounded case), a well known and very useful result which states, in rough terms, that if financial markets are free of arbitrage opportunities, then there exists a probability measure with respect to which asset returns are martingales. The existing versions of this theorem differ from one another in the class of admissible trading strategies considered and in the different definitions of an arbitrage opportunity adopted. In many papers the latter concept is strengthened into that termed free lunch, whose definition is directly inspired by the condition originally proposed by Yan (and, independently, by Kreps[13]) and requires the choice of a reference topological space. In [1], the L^p definition is adopted relatively to simple investment strategies whereas in [6] free lunches are defined with reference to L^∞ (so-called *free-lunches-with-vanishing-risk*), but the full fledge of stochastic integration is exploited by admitting general investment strategies. [7] and [14] focus

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on arbitrage opportunities with continuous price process and general integrands as portfolios.

The main content of the present work may be summarised as follows. In Section 3 we apply our version of the Yan theorem to a model of financial markets in which only simple investment strategies are admitted but free lunches, defined with reference to L^∞, are ruled out. This makes our results akin to those established in [6, Section 7] (see the more detailed comments below). It should be remarked that the absence of free lunches in L^∞ represents a much weaker constraint on markets than the corresponding condition formulated in the L^p framework. It therefore guarantees less stringent mathematical properties, which explains the interest for a corresponding version of Yan characterization. In fact we prove that the absence of free lunches implies the existence of a strictly positive local martingale Z that, if adopted as a discount factor, transforms asset prices and returns into local martingales. We also prove that, when focusing on arbitrage opportunities rather than free lunches, the same conclusion follows save that the intervening discount factor need not be strictly positive. In either case, however, the mere existence of the process Z does not provide a sufficient condition for excluding arbitrage opportunities. Finally in Section 4 we remark on the financial interpretation of the results obtained and the conclusion will be given in Section 5.

2 Yan Theorem

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space and $\mathfrak{M}$ the space of bounded, finitely additive measures on $\mathcal{F}$ vanishing on P null sets (usually denoted by $ba(\Omega, \mathcal{F}, P)$, as in [8]). By $f \in L^\infty_{++}$ we mean $f \in L^\infty_+$ and $P(f) > 0$. We shall speak of a *strictly positive* measure m (denoted by $m \in \mathfrak{M}_{++}$) if m is a positive set function (denoted by $m \in \mathfrak{M}_+$) and $m(f) > 0$ for any $f \in L^\infty_{++}$. Since $\mathfrak{M}$ is the topological dual of L^∞ (see [8, Theorem IV.8.16]), we denote by τ the weak* topology on $\mathfrak{M}$ and denote by $\mathcal{M}^\tau$ the τ closure of any subset $\mathcal{M}$ of $\mathfrak{M}$. We also find it convenient not to distinguish between a set and its indicator in notations.

We recall the decomposition of Yosida and Hewitt (see [24, Theorem 1.24] or [3, Theorem 10.2.1]): for each $m \in \mathfrak{M}_+$ there exists a unique way of writing $m = m^c + m^\perp$ with $m^c, m^\perp \in \mathfrak{M}_+$, m^c is countably additive and absolutely continuous with respect to P and $m^\perp$ is purely finitely additive, i.e. for any $\varepsilon > 0$, there exists $F \in \mathcal{F}$ such that $m^\perp(F) = 0$ and $P(F^c) < \varepsilon$ ([24, see Theorem 1.19] or [3, Theorem 10.3.3]). We remark that if $m \in \mathfrak{M}_+$ and $F \in \mathcal{F}$ such that $P(F) > 0 = m^c(F)$, then by orthogonality we can find $F' \subset F$, $F' \in \mathcal{F}$ such that $P(F') > P(F)(1-\varepsilon)$ and $m(F') = 0$. In other words, if $m \in \mathfrak{M}_{++}$, then m^c is equivalent to P; it is obvious that the converse is also true.

We start this section with the following theorem which is a variant of the Farkas lemma and therefore may be of interest on its own.

Theorem 1. *Let $\mathcal{M}$ be a convex subset of $\mathfrak{M}_+$ which is relatively τ compact. Then either one of the following mutually exclusive properties holds:*

(i). $P(f) > 0 = \sup\{m(f) : m \in \mathcal{M}\}$ for some $f \in L^\infty_{++}$;

(ii). $\mathcal{M}^\tau$ admits a strictly positive element.

Proof. (ii) contradicts (i) as $\sup\{m(f) : m \in \mathcal{M}\} = \sup\{m(f) : m \in \mathcal{M}^\tau\}$ for each $f \in L^\infty$. Thus we only need to prove that (i) holds when (ii) fails. The τ topology makes $\mathcal{M}$ a Hausdorff, locally convex, topological vector space (see [20, Proposition 21, p.240]).

For $m \in \mathfrak{M}$, let P_m be the component of P orthogonal to m^c in the Lebesgue decomposition of P and denote $\mathcal{S}(m) = \{F \in \mathcal{F} : P_m(F^c) = m^c(F) = 0\}$ and

$$\eta = \inf\{P(F) : F \in \mathcal{S}(m),\ m \in \mathcal{M}^\tau\}$$

Let $\langle m_r \rangle_{r\in\mathbb{N}}$ and $\langle F_r \rangle_{r\in\mathbb{N}}$ be sequences in $\mathcal{M}^\tau$ and $\mathcal{F}$ respectively such that $F_r \in \mathcal{S}(m_r)$ for $r \geq 1$ and $\eta = \lim_r P(F_r)$. Define $F_0 = \cap_r F_r$ and $m_0 = \sum_r 2^{-r} m_r$: then $P(F_0) \leq \eta$. If

$G_r \in \mathcal{F}$, $m_r^\perp(G_r) = 0$ and $P(G_r^c) < \varepsilon 2^{-r}$ for $r \geq 1$ and if we set $G = \cap_r G_r$, then $P(G^c) \leq \varepsilon$ while $\sum_r 2^{-r} m_r^\perp(G) = 0$: this proves that $\sum_r 2^{-r} m_r^\perp \in \mathfrak{M}_+$ is purely finitely additive. Since $\sum_r 2^{-r} m_r^c \in \mathfrak{M}_+$ is countably additive and the Yosida and Hewitt decomposition is unique, we conclude that $m_0^c = \sum_r 2^{-r} m_r^c$. Clearly, $m_0^c(F_0) = 0$. If $E \in \mathcal{F}$ and $m_0^c(E) = 0$, then $m_r^c(E) = 0$ for each r so that $P(EF_0^c) \leq \sum_r P(EF_r^c) = 0$. In other words $P \ll m_0^c$ in restriction to F_0^c so that $P_{m_0}(F_0^c) = 0$ or, equivalently, $F_0 \in \mathcal{S}(m_0)$. However, since $\mathcal{M}^\tau$ is convex and closed, $m_0 \in \mathcal{M}^\tau$ and it then follows that $P(F_0) \geq \eta$. We have thus shown that η is actually attained so that if (ii) fails then $\eta > 0$.

Let $m \in \mathcal{M}^\tau$ and $n \in \mathbb{N}$. Since $m^\perp$ and P are orthogonal, there exists a $\mathcal{F}$ measurable subset F_m^n of $F_m \in \mathcal{S}(m)$ such that $m(F_m^n) = 0$ and $P(F_m^n) > \eta(1 - 2^{-n})$ and by the axiom of choice we obtain a collection $\{F_m^n : m \in \mathcal{M}^\tau, n \in \mathbb{N}\}$ of sets with this property. Define the set

$$\mathcal{U}_m^n = \{m' \in \mathcal{M}^\tau : m'(F_m^n) < 2^{-n}\}.$$

As $\mathcal{U}_m^n$ contains m, $\{\mathcal{U}_m^n : m \in \mathcal{M}^\tau\}$ is an open cover of the compact set $\mathcal{M}^\tau$. There exists then a finite collection $\{\varphi_i : i = 1, \cdots, I\}$ of continuous maps $\varphi_i : \mathcal{M}^\tau \to [0,1]$ each vanishing outside $\mathcal{U}_{m_i}^n$ for some $m_i \in \mathcal{M}^\tau$ and such that $\sum_{i=1}^{I} \varphi_i(m) = 1$ for each $m \in \mathcal{M}^\tau$ [20, proposition 16, p. 200]. Define the functions $h_n : \mathcal{M}^\tau \to L_+^\infty$ and $\rho_n : \mathcal{M}^\tau \times \mathcal{M}^\tau \to [0,1]$ implicitly as

$$h_n(m) = \sum_{i=1}^{I} \varphi_i(m) F_{m_i}^n \quad \text{and} \quad \rho_n(m', m) = m'\big(h_n(m)\big).$$

It is immediate that h_n is continuous and therefore so is $m \to \rho_n(m', m)$; moreover, $m' \to \rho_n(m', m)$ is linear. By a theorem of Ky Fan (see [9, Theorem 1]), it follows that there exists $m_n \in \mathcal{M}^\tau$ such that

$$\sup_{m \in \mathcal{M}^\tau} \rho_n(m, m_n) \leq \sup_{m \in \mathcal{M}^\tau} \rho_n(m, m).$$

However, by construction if $m \in \mathcal{M}^\tau$ and $\varphi_i(m) > 0$, then $m \in \mathcal{U}_{m_i}^n$ i.e. $m(F_{m_i}^n) < 2^{-n}$ so that

$$\rho_n(m, m) = \sum_{i=1}^{I} \varphi_i(m) m(F_{m_i}^n) < 2^{-n}.$$

Let $h_n = h_n(m_n)$. We have thus obtained a sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ such that $m(h_n) < 2^{-n}$ for every $m \in \mathcal{M}^\tau$ while

$$P(h_n) = \sum_{i=1}^{I} \varphi_i(m_n) P(F_{m_i}^n) \geq \sum_{i=1}^{I} \varphi_i(m_n) \eta(1 - 2^{-n}) = \eta(1 - 2^{-n}).$$

Replacing h_n with an appropriate convex combination $f'_n = \sum_{k=0}^{K_n} \alpha_k^n h_{n+k}$, we obtain, by the Komlos lemma (see [6, Lemma A1.1]), that the sequence $\langle f'_n \rangle_{n \in \mathbb{N}}$ admits a P a.s. limit f' i.e., by the Egoroff theorem (see [8, theorem II.6.12]), that it converges to f' *uniformly* outside some $F \in \mathcal{F}$ such that $P(F^c) < \eta\delta$, for δ arbitrarily small. Let $f_n = f'_n F$ and $f = f'F$. Given that $0 \leq f_n \leq f'_n \leq 1$, then for $m \in \mathcal{M}^\tau$,

$$m(f) = \lim_n m(f_n) \leq \liminf_n m(f'_n) = \liminf_n \sum_{k=0}^{K_n} \alpha_k^n m(h_{n+k}) < \lim_n 2^{-n} = 0$$

while

$$P(f) = \lim_n P(f_n) \geq \lim_n P(f'_n) - P(F^c) = \lim_n \sum_{k=0}^{K_n} \alpha_k^n P(h_{n+k}) - P(F^c) \geq \eta(1 - \delta)$$

and (i) follows.

Theorem 1 may be restated by saying that if the convex sets $\mathcal{M}^\tau$ and $\mathfrak{M}_{++}$ are disjoint, then they may be separated via a linear functional which is not only τ continuous but strictly positive as well. The interest for this conclusion is that in the general case $\mathfrak{M}_{++}$ will neither be closed nor contain an interior point. So the claim is somewhat stronger than the usual separation theorems[1].

A typical example of Theorem 1 arises when the set $\mathcal{M}$ consists of finitely additive probabilities separating convex subsets of L^∞. In those situations, the most interesting one is the L^∞ version of the theorem of Yan in which $\mathcal{K} \subset L^\infty$ is a convex set containing the origin, $\mathcal{C} = \mathcal{K} - L^\infty_+$ and $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in the norm topology of L^∞. Let also

$$\mathcal{M}_\mathcal{K} = \left\{m \in \mathfrak{M}_+ : m(\Omega) = 1, \ \sup_{k\in\mathcal{K}} m(k) < \infty\right\}$$

and

$$\mathcal{M}^1_\mathcal{K} = \{m \in \mathfrak{M}_+ : m[\overline{\mathcal{C}}] \le 1, \|m\| \le 1\}$$

Theorem 2. *The following statements are equivalent:*

1. *for every $f \in L^\infty_{++}$ there exists $c > 0$ such that $cf \notin \overline{\mathcal{C}}$;*
2. *for every $F \in \mathcal{F}$ such that $P(F) > 0$, there exists $c > 0$ such that $cF \notin \overline{\mathcal{C}}$;*
3. *$\mathcal{M}_\mathcal{K}$ admits a strictly positive element.*

Proof. $(3 \to 1)$. Let m be a strictly positive element of $\mathcal{M}_\mathcal{K}$, $f \in L^\infty_{++}$ and $c > 0$ be such that $cf \in \overline{\mathcal{C}}$: $\sup_{k\in\mathcal{K}} m(k) = \sup_{x\in\overline{\mathcal{C}}} m(x) \ge cm(f) > 0$. Therefore, if 1 fails so does 3. The implication $(1 \to 2)$ is obvious.

$(2 \to 3)$. Let $cF \notin \overline{\mathcal{C}}$ and ϕ_F be a continuous, non trivial linear functional on L^∞ separating $\{cF\}$ and $\overline{\mathcal{C}}$ (see [8, Corollary V.2.11]). Let also $\overline{m}_F$ be the element of $\mathfrak{M}$ representing ϕ_F. $0 \in \mathcal{K}$ implies that $\overline{m}_F[\overline{\mathcal{C}}] < a < c\overline{m}_F(F)$ for some $a > 0$; $-L^\infty_+ \subset \overline{\mathcal{C}}$ implies that $\overline{m}_F[-L^\infty_+]$ is a convex cone in $(-\infty, a)$ i.e. $\overline{m}_F[L^\infty_+] \ge 0$ so that $\overline{m}_F \in \mathfrak{M}_+$ and $\overline{m}_F(\Omega) > 0$ (as ϕ_F is non trivial). Letting $m_F = [(1+c)\|\overline{m}_F\|]^{-1}\overline{m}_F$, we conclude that $m_F \in \mathcal{M}^1_\mathcal{K}$ and $m_F(F) > 0$. Thus $\mathcal{M}^1_\mathcal{K}$ is non empty, convex and τ compact [8, Lemma I.5.7(a), p. 17 and theorem V.4.2, p. 424]. Moreover, it fails to possess Property (i) of Theorem 1 and admits, as a consequence, a strictly positive element $\overline{m}$. The claim is established with $m = \|\overline{m}\|^{-1}\overline{m}$ in place of $\overline{m}$.

The L^p, $1 \le p < \infty$ versions of this theorem considered by Yan[23] and by Ansel and Stricker[1] rely crucially on the fact that in that framework separating measures admit a density, a property that does not carry over to $\mathfrak{M}$ as the Radon Nikodym theorem fails in the absence of countable additivity. The minimax inequality exploited in the proof of Theorem 1 allows us to overcome such difficulty.[2]

3 Applications to Mathematical Finance

Let $S = (S_t : t \in \mathbb{R}_+)$ be a continuous, $\mathbb{R}^d$ valued process over the probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t : t \in \mathbb{R}_+)$ satisfying the usual assumptions of completeness and right continuity and, without loss of generality, assume $\mathcal{F} = \sigma(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t)$. Denote by $\mathcal{T}$ the set of all stopping times on the underlying filtration and, if $\tau \in \mathcal{T}$, let $\mathcal{T}_\tau = \{\upsilon \in \mathcal{T} : P(\upsilon > \tau) = 1\}$. S represents asset prices in discounted units. Let Θ be the set of $\mathbb{R}^d$ valued, simple processes

[1] An attempt to obtain a version of this theorem with L^∞ replaced by the space of bounded functions $\mathfrak{B}(\mathcal{F})$ was made in [4, Lemma A.6].

[2] After this paper was completed, I came across the work of Rohklin[19] in which a version of Thereom 2 is proved for the special case in which $\mathcal{C}$ is a convex cone by convex duality methods.

θ such that $l_\theta \equiv \sup_{t\in\mathbb{R}_+} |\int_0^t \theta dS| \in L^\infty$. Write for simplicity $K^\theta = (\int_0^t \theta dS : t \in \mathbb{R}_+)$, the process describing the (discounted) returns of the investment strategy $\theta \in \Theta$ and $K(\Theta) = \{K^\theta : \theta \in \Theta\}$. Observe that, by continuity, each of the components of the vector valued process S is locally in $K(\Theta)$. It should be remarked that the definition of the stochastic integral $\int \theta dS$ is necessarily limited to the case in which θ is a simple process, unless one is prepared to make more stringent assumptions on the nature of the price process S. Define the sets

$$\mathcal{K} = \{K_\infty^\theta : \theta \in \Theta\} \text{ and } \mathcal{C} = \mathcal{K} - L_+^\infty.$$

Assume that there are no free lunches in the sense initially introduced by Delbaen and Schachermayer[6], i.e.

$$\overline{\mathcal{C}} \cap L_+^\infty = \{0\} \tag{1}$$

A weaker notion is that of absence of arbitrage opportunities defined as

$$\mathcal{C} \cap L_+^\infty = \{0\} \tag{2}$$

Of course, since $0 \in \mathcal{K}$, then Theorem 2 establishes that if (1) holds, there exists a strictly positive $m \in \mathcal{M}_{\mathcal{K}}$: this clearly implies that $m[\overline{\mathcal{C}}] \le 0$ and $m[\mathcal{C}_0] = 0$ for any linear subspace $\mathcal{C}_0$ of $\mathcal{C}$ – such as $\mathcal{K}$. On the other hand, if (2) holds, then given that L_{++}^∞ has an internal point – e.g. Ω –, we conclude that (see [8, Theorem V.2.8])there exists a non null element $m \in \mathfrak{M}$ such that $m[\mathcal{C}] \le 0 \le m[L_+^\infty]$ and that can therefore be normalized to be a finitely additive probability. For the rest of this section m will be fixed.

The application of the Yan theorem to financial modelling is therefore related to the absence of free lunches. It should be remarked that, distinct from the L^p case treated in [1], the L^∞ definition of a free lunch introduced by Delbaen and Schachermayer did not foster the proof of a corresponding version of the Yan theorem. In fact, given the extended set of trading strategies considered in [6], available upon assuming the semimartingale nature of the price process, the norm and the weak* closure of $\mathcal{C}$ turn out to be equivalent in the absence of free lunches. In a less perfect market, such as the one considered here, this remarkable property fails so that Theorem 2 above gains importance. The issue now is to show that, despite finite additivity, it is still possible to obtain a nice and tractable pricing rule.

Let $\tau \in \mathcal{T}$ and denote by m_τ the restriction of m to $\mathcal{F}_\tau$ and by $m_\tau^c + m_\tau^\perp$ its Yosida and Hewitt decomposition. Since m_τ^c coincides with the restriction to $\mathcal{F}_\tau$ of the outer measure generated by m_τ (see [3, Theorem 10.2.2]), (i.e. $m_\tau^c(F) = \inf\{\sum_n m(F_n) : F_n \in \mathcal{F}_\tau,\ F = \bigcup_n F_n\}$) and given that $F\{\tau \le \upsilon\} \in \mathcal{F}_\upsilon$ when $\upsilon \in \mathcal{T}$ and $F \in \mathcal{F}_\tau$, we conclude

$$\begin{aligned} m_\upsilon^c(F; \tau \le \upsilon) &= \inf\Big\{\sum_n m(G_n) : G_n \in \mathcal{F}_\upsilon,\ F\{\tau \le \upsilon\} = \bigcup_n G_n\Big\} \\ &\le \inf\Big\{\sum_k m(F_k; \tau \le \upsilon) : F_n \in \mathcal{F}_\tau,\ F = \bigcup_k F_k\Big\} \\ &= m_\tau^c(F; \tau \le \upsilon). \end{aligned} \tag{3}$$

Analogously, $m_\upsilon^\perp(F; \tau \le \upsilon) \ge m_\tau^\perp(F; \tau \le \upsilon)$. Furthermore, if $\upsilon \in \mathcal{T}_\tau$

$$(m_\tau^c - m_\upsilon^c)(F) = (m_\tau - m_\upsilon)(F) + (m_\upsilon^\perp - m_\tau^\perp)(F) = (m_\upsilon^\perp - m_\tau^\perp)(F). \tag{4}$$

In order to take care of the rather delicate issue of "regularity" we introduce a modified decomposition of m_τ. To this end, note that $\mathcal{T}_\tau$ is a directed set if we let $\upsilon' \succsim \upsilon$ whenever $P(\upsilon' \le \upsilon) = 1$. We can define the set functions

$$m_{\tau+}^c = \lim_{\upsilon \in \mathcal{T}_\tau} m_\upsilon^c | \mathcal{F}_\tau \quad \text{and} \quad m_{\tau+}^\perp = \lim_{\upsilon \in \mathcal{T}_\tau} m_\upsilon^\perp | \mathcal{F}_\tau, \tag{5}$$

so that $m_\tau = m^c_{\tau+} + m^\perp_{\tau+}$. (5) implies that $m^c_{\tau+}$ and $m^\perp_{\tau+}$ are positive set functions on $\mathcal{F}_\tau$ and by (3), we have

$$m^c_\tau(F) \geq m^c_{\tau+}(F) \geq m^c_\upsilon(F) \geq m^c(F) \tag{6}$$

for each $F \in \mathcal{F}_\tau$ and $\upsilon \in \mathcal{T}_\tau$. $m^c_{\tau+}$ is then countably additive and absolutely continuous with respect to P. Let $Y_\tau \in L^1(\Omega, \mathcal{F}_\tau, P)_+$ be the corresponding Radon Nikodym derivative. We state in the following lemma some of its useful properties.

Lemma 1. *Let $Y = (Y_t : t \in \mathbb{R}_+)$ and $m^c_{\tau+}$ be defined as above.*

(i). Y is a positive supermartingale admitting a right continuous modification;

(ii). If there are no free lunches i.e. (2) holds and if $Y_\infty = \lim_t Y_t$, P a.s. then $P(Y_\infty > 0) = 1$;

(iii). For any sequence $\langle \tau_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $\tau_{n+1} \in \mathcal{T}_{\tau_n}$, $\sum_n \| m^c_{\tau_n+} - m^c_{\tau_n} \| < 1$.

Proof. Let $F \in \mathcal{F}_t$, $t < u$ and $\upsilon \in \mathcal{T}_u$. By (6), $m^c_{t+}(F) \geq m^c_\upsilon(F)$ i.e. $m^c_{t+}(F) \geq m^c_{u+}(F)$ and it follows that Y is a positive supermartingale and admits an a.s. limit Y_∞ by Doob's limit theorem. If $\tau \in \mathcal{T}_t$, then by (3)

$$m^c_\tau(\Omega) \leq m^c_{t+2^{-n}}(\tau > t + 2^{-n}) + m^c_\tau(\tau \leq t + 2^{-n}) \leq \lim_{u>t, u\downarrow t} m^c_{u+}(\Omega) + m^c_\tau(\tau \leq t + 2^{-n}).$$

Since $\lim_n m^c_\tau(\tau \leq t + 2^{-n}) = 0$, we conclude that $m^c_{t+}(\Omega) \leq \lim_{u>t, u\downarrow t} m^c_{u+}(\Omega)$. In other words, the function $t \to P(Y_t)$ is right continuous so that Y admits a right continuous modification by virtue of a fundamental result of Meyer[16,VI,T4,p.95]. Let y be the Radon Nikodym derivative of m^c with respect to P: under (1), $P(y = 0) = 0$. By (3) and martingale convergence we obtain the inequality

$$Y_\infty = \lim_t Y_t \geq \lim_t P(y|\mathcal{F}_t) = y,$$

from which the second claim readily follows. Let $\langle \tau_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{T}$ with $\tau_{n+1} \in \mathcal{T}_{\tau_n}$. Then $m^c_{\tau_n+} \leq m^c_{\tau_n}$ and $m^c_{\tau_n+}(\Omega) \geq m^c_{\tau_{n+1}}(\Omega)$ so that

$$\sum_n \| m^c_{\tau_n+} - m^c_{\tau_n} \| = \sum_n (m^c_{\tau_n} - m^c_{\tau_n+})(\Omega) \leq \sum_n (m^c_{\tau_n} - m^c_{\tau_{n+1}})(\Omega) \leq m^c_0(\Omega) \leq 1.$$

Without loss of generality we assume that Y has right continuous paths. Let $Y = M - A$ be the semimartingale decomposition of Y, where M is a positive local martingale and A is an increasing, predictable, integrable process. Denote by $\mathcal{T}_Y$ the set of stopping times τ such that the stopped process Y^τ is of class D.

We proceed now to the explicit construction of a class of return processes. To this end, fix $\theta \in \Theta$ and define the sequence $\mathcal{U}^n = \langle \upsilon^n_k \rangle_{k\in\mathbb{N}}$ of stopping times by letting $\upsilon^n_0 = 0$ and, for $k \geq 1$,

$$\upsilon^n_k = \inf\{t > \upsilon^n_{k-1} : |K^\theta_t - K^\theta_{\upsilon^n_{k-1}}| \geq 2^{-n} \text{ or } t > \upsilon^n_{k-1} + 2^{-n}\}.$$

$\mathcal{U}^n$ is clearly an adapted subdivision and $\langle \mathcal{U}^n \rangle_{n\in\mathbb{N}}$ a Riemann sequence, according to the terminologies in [10, p.51]. As $\langle \upsilon^n_k \rangle_{k\in\mathbb{N}}$ increases P a.s. to ∞, let I_n be an integer such that $P(\upsilon^n_{I_n} > 2^n) > 1 - 2^{-n}$.

For each $k \geq 0$, let $F^n_k \in \mathcal{F}_{\upsilon^n_k}$ so that (i) $F^n_{k+1} \subset F^n_k$, (ii) $m^\perp_{\upsilon^n_k}(F^n_k) = 0$ and (iii) $P(F^n_k) > 1 - \frac{2^{-n}}{1+2^{-k}}$ (so that $P(\bigcap_k F^n_k) \geq 1 - 2^{-n}$). Let

$$\mathcal{C}_\theta = \Big\{ a \sum_{k=1}^{I} F^n_{k-1}(K^\theta_{\upsilon^n_k \wedge \tau} - K^\theta_{\upsilon^n_{k-1} \wedge \tau}) : I, n \in \mathbb{N}, a \in \mathbb{R}, \tau \in \mathcal{T}_Y \Big\}. \tag{7}$$

Consider for the moment $\tau \in \mathcal{T}_Y$ as fixed and write τ_k^n for $\upsilon_k^n \wedge \tau$. Observe that $\{F_{k-1}^n; \upsilon_{k-1}^n < \tau\} \in \mathcal{F}_{\tau_{k-1}^n}$ and that $m_{\tau_{k-1}^n}^{\perp}(F_{k-1}^n; \upsilon_{k-1}^n < \tau) \le m_{\upsilon_{k-1}^n}^{\perp}(F_{k-1}^n) = 0$. Therefore by (4),

$$\begin{aligned}
|m_{\tau_k^n+}^{\perp}(F_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta))| \le & 2^{-n} m_{\tau_k^n+}^{\perp}(F_{k-1}^n; \upsilon_{k-1}^n < \tau) \\
= & 2^{-n}(m_{\tau_k^n+}^{\perp} - m_{\tau_{k-1}^n}^{\perp})(F_{k-1}^n; \upsilon_{k-1}^n < \tau) \\
= & 2^{-n}(m_{\tau_{k-1}^n}^{c} - m_{\tau_k^n+}^{c})(F_{k-1}^n; \upsilon_{k-1}^n < \tau) \\
\le & 2^{-n}[(m_{\tau_{k-1}^n+}^{c} - m_{\tau_k^n+}^{c})(F_{k-1}^n) + \|m_{\tau_{k-1}^n+}^{c} - m_{\tau_{k-1}^n}^{c}\|] \\
= & 2^{-n}[P((Y_{\tau_{k-1}^n} - Y_{\tau_k^n})F_{k-1}^n) + \|m_{\tau_{k-1}^n+}^{c} - m_{\tau_{k-1}^n}^{c}\|].
\end{aligned}$$

While on the other hand,

$$m_{\tau_k^n+}^{c}\big(F_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta)\big) = P\big(Y_{\tau_k^n} F_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta)\big).$$

As a consequence, if $|a| = 1$,

$$\begin{aligned}
m\big(aF_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta)\big) = & (m_{\tau_k^n+}^{c} + m_{\tau_k^n+}^{\perp})\big(aF_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta)\big) \\
\ge & aP\big(Y_{\tau_k^n} F_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta)\big) \\
& - 2^{-n}\big[P\big((Y_{\tau_{k-1}^n} - Y_{\tau_k^n})F_{k-1}^n\big) + \|m_{\tau_{k-1}^n+}^{c} - m_{\tau_{k-1}^n}^{c}\|\big].
\end{aligned}$$

Since the sequence $\langle F_k^n \rangle_{k\in\mathbb{N}}$ is decreasing, we further establish that

$$\begin{aligned}
\sum_{k=1}^{I_n}(Y_{\tau_{k-1}^n} - Y_{\tau_k^n})F_{k-1}^n = & \sum_{k=1}^{I_n-1}(Y_0 - Y_{\tau_k^n})F_{k-1}^n F_k^{nc} + (Y_0 - Y_{\tau_{I_n}^n})F_{I_n-1}^n \\
\le & Y_0(F_{I_n-1}^n \cup \bigcup_{k=1}^{I_n-1} F_{k-1}^n F_k^{nc}) \\
\le & Y_0,
\end{aligned}$$

i.e. $\sum_{k=1}^{I_n}\{(Y_{\tau_{k-1}^n} - Y_{\tau_k^n})F_{k-1}^n + \|m_{\tau_{k-1}^n+}^{c} - m_{\tau_{k-1}^n}^{c}\|\} \le (Y_0 + 1)$ by Lemma 1 (iii). Given that $\mathcal{C}_\theta$ is a convex cone in $\mathcal{C}$ and in view of the preceding developments we conclude that

$$0 \ge m\Big(\sum_{k=1}^{I_n} aF_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta)\Big) \ge aP\sum_{k=1}^{I_n} Y_{\tau_k^n} F_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta) - 2^{-n}[P(Y_0) + 1].$$

In other words,

$$0 = \lim_n P\sum_{k=1}^{I_n} Y_{\tau_k^n} F_{k-1}^n(K_{\tau_k^n}^\theta - K_{\tau_{k-1}^n}^\theta). \tag{8}$$

Lemma 2. *Let $\sigma \in \mathcal{T}$. Then*

$$0 = P\Big\{Y_\sigma^\tau K_{\tau\wedge\sigma}^\theta + \int_0^{\tau\wedge\sigma} K^\theta dA\Big\}. \tag{9}$$

Proof. Given that $K_0^\theta = 0$ (by definition), the sum appearing in (8) may be rewritten as

$$Y_{\tau_{I_n}^n} K_{\tau_{I_n}^n}^\theta F_{I_n-1}^n + \sum_{k=1}^{I_n-1} Y_{\tau_k^n} K_{\tau_k^n}^\theta F_{k-1}^n F_k^{nc} - \sum_{k=1}^{I_n} F_{k-1}^n K_{\tau_{k-1}^n}^\theta (Y_{\tau_k^n} - Y_{\tau_{k-1}^n}).$$

Furthermore, $|P(Y_{\tau_k^n} K^\theta_{\tau_k^n} F^n_{k-1} F^{nc}_k)| \leq l_\theta P(M_\tau F^n_{k-1} F^{nc}_k)$ and $|P\big(F^{nc}_{k-1} K^\theta_{\tau^n_{k-1}}(Y_{\tau_k^n} - Y_{\tau^n_{k-1}})\big)| \leq l_\theta P\big(F^{nc}_{I_n-1}(A_{\tau_k^n} - A_{\tau^n_{k-1}})\big)$ imply that

$$\Big|P\sum_{k=1}^{I_n-1} Y_{\tau_k^n} K^\theta_{\tau_k^n} F^n_{k-1} F^{nc}_k\Big| \leq l_\theta P\Big(M_\tau \bigcup_{k=1}^{I_n-1} F^n_{k-1}F^{nc}_k\Big) \leq l_\theta P(M_\tau F^{nc}_{I_{n-1}})$$

and

$$\Big|P\sum_{k=1}^{I_n} F^{nc}_{k-1} K^\theta_{\tau^n_{k-1}}(Y_{\tau_k^n} - Y_{\tau^n_{k-1}})\Big| \leq l_\theta P\Big(F^{nc}_{I_n-1}\sum_{k=1}^{I_n}(A_{\tau_k^n} - A_{\tau^n_{k-1}})\Big) \leq l_\theta P(M_\tau F^{nc}_{I_n-1})$$

respectively. Given that the sequence $\langle Y_{\tau^n_{I_n}} K^\theta_{\tau^n_{I_n}} F^n_{I_n-1}\rangle_{n\in\mathbb{N}}$ is uniformly integrable $\lim_n P(M_\tau F^{nc}_{I_n-1}) = 0$ and the bounded convergence of the stochastic integral (see [10, Theorem I.4.31 (iii)]), (8) is translated into

$$\begin{aligned} 0 =& \lim_n P\Big\{Y_{\tau^n_{I_n}} K^\theta_{\tau^n_{I_n}} F^n_{I_n-1} - \sum_{k=1}^{I_n} K^\theta_{\tau^n_{k-1}}(Y_{\tau^n_k} - Y_{\tau^n_{k-1}})\Big\} \\ =& \lim_n P\Big\{Y_{\tau^n_{I_n}} K^\theta_{\tau^n_{I_n}} F^n_{I_n-1} + \sum_{k=1}^{I_n} K^\theta_{\tau^n_{k-1}}(A_{\tau^n_k} - A_{\tau^n_{k-1}})\Big\} \\ =& P\Big\{Y_\tau K^\theta_\tau + \int_0^\tau K^\theta dA\Big\}, \end{aligned}$$

while (9) follows from the fact that $\sigma \wedge \tau \in \mathcal{T}_Y$ whenever $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}_Y$.

The process $YK^\theta + \int K^\theta dA$ is *càdlàg*, starts at 0 and, upon stopping at τ, satisfies (9) for any stopping time σ: by Lemma 1.44 in [10], it is a local martingale and therefore YK^θ a semimartingale. Since Y is strictly positive the process Y^{-1} is well defined. Given that the inverse function is convex over the set $]0,\infty[$, it is itself a semimartingale (see [18, Theorem VI.1.1]). Then so is K^θ as it is the product of two semimartingales. Let $M^\theta + V^\theta$ be the semimartingale decomposition of K^θ – with V^θ predictable and of locally integrable variation and M^θ a local martingale.

Exploiting the semimartingale nature of K^θ and integration by parts, we obtain

$$YK^\theta + \int K^\theta dA - \int K^\theta dM - \int Y_- dM^\theta = \int Y_- dV^\theta + \langle K^\theta, Y^c\rangle. \tag{10}$$

It follows from (9) that in (10) the local martingale appearing on the left hand side is predictable while that on the right hand side is of locally integrable variation. Therefore the process $V^\theta + \langle K^\theta, \int Y_-^{-1} dM^c\rangle$ must vanish, by the uniqueness of the Doob Meyer decomposition. Define the local martingale

$$Z = \mathcal{E}\int Y_-^{-1} dM, \tag{11}$$

where $\mathcal{E}$ is the exponential martingale of Doléans-Dade ($\mathcal{L}$ denotes the stochastic logarithm).

The preceding discussions may be restated in the Proposition that follows. The following definition is borrowed from [21].

Definition 1. *A stochastic process X is a martingale density for S if X is a positive local martingale starting at $X_0 = 1$ and such that XS is a local martingale; X is strictly positive if $P(X_\infty > 0) = 1$.*

In the realm of finance (see [2], among others), sometimes a martingale density for the discounted price process S is termed *stochastic discount factor* or *market price of risk*.

Proposition 1. *The absence of free lunches, as defined by (1), implies that the price process S is a semimartingale and there exists a strictly positive martingale density for S.*

Proof. S is a semimartingale since K^θ is a semimartingale for each $\theta \in \Theta$ and as remarked above, the components of S are locally in $K(\Theta)$. By construction, Z is a positive local martingale starting at $Z_0 = 1$. To show that $\{Z_\infty = 0\} \subset \{Y_\infty = 0\}$, recall that $\langle U^c, A^c\rangle = 0$ for any semimartingale U (see [10, 4.49.(d)]) and that $M = Y + A$. We then obtain from (11) that, for each $t \in \mathbb{R}_+ \cup \{\infty\}$,

$$
\begin{aligned}
Z_t =& \exp\Big\{\int_0^t Y_-^{-1} dM - \frac{1}{2}\int_0^t Y_-^{-2} d\langle M^c, M^c\rangle\Big\} \prod_{s\le t} e^{-Y_{s-}^{-1}\Delta M_s}(1 + Y_{s-}^{-1}\Delta M_s) \\
\ge& \exp\Big\{\int_0^t Y_-^{-1} dM - \frac{1}{2}\int_0^t Y_-^{-2} d\langle M^c, M^c\rangle\Big\} \prod_{s\le t} e^{-Y_{s-}^{-1}\Delta M_s}(1 + Y_{s-}^{-1}\Delta Y_s) \\
=& \exp\Big\{\int_0^t Y_-^{-1} dY - \frac{1}{2}\int_0^t Y_-^{-2} d\langle Y^c, Y^c\rangle + \int_0^t Y_-^{-1} dA^c\Big\} \prod_{s\le t} e^{-Y_{s-}^{-1}\Delta Y_s}(1 + Y_{s-}^{-1}\Delta Y_s) \\
=& Y_t \exp\Big\{\int_0^t Y_-^{-1} dA^c\Big\} \\
\ge& Y_t.
\end{aligned} \tag{12}
$$

Therefore if (1) holds, then Z is a strictly positive local martingale by Lemma 1 (ii). To see that Z is a martingale density we just perform once again integration by parts and obtain $K^\theta Z = \int K^\theta dZ + \int Z_- dM^\theta$.

Proposition 1 should be compared to Theorem 7.6(a) in [6] where it is claimed that the absence of free lunches for simple integrands and over a bounded time interval implies the semimartingale property and the absence of free lunches for *general* integrands. This means the existence of a probability measure Q equivalent to P transforming asset returns into local martingales[3]. To strengthen this analogy, note that in restriction to any *stochastic* interval $[[0, \tau]]$ where $\tau \in \mathcal{T}_Z$ (i.e. Z^τ is of class D), there exists an equivalent local martingale measure, simply defined as $dQ_\tau = Z_\tau dP$. However, we obtain the additional conclusion that if $\upsilon \in \mathcal{T}_\tau \cap \mathcal{T}_Z$, then the corresponding local martingale measure Q_υ relative to $[[0, \upsilon]]$ satisfies $.Q_\upsilon|\mathcal{F}_\tau = Q_\tau$.

It is an open question what is the “ right” definition of an arbitrage opportunity. Although in the preceding proposition we considered the property that markets do not admit free lunches, the definition of an arbitrage opportunity implicit in (2) is definitely more sound in economic terms as it does not involve limit points of investment profits. We have already remarked that under (2) it is still possible to recover a separating, finitely additive probability measure m. In general this will not be strictly positive so that, letting Y have the same meaning as above, it is useful to define the stopping time

$$T = \inf\{t : Y_t = 0\}$$

An open issue is the assessment of the probability of the event $\{T < \infty\}$.

Corollary 1. *The absence of arbitrage opportunities, as defined by (2), implies that the price process S stopped at T is a semimartingale which admits a martingale density.*

Proof. It is clear that (9) was derived without any reference to Y being strictly positive. We thus still deduce from (9) that YK^θ is a semimartingale as well as the stopped process K^{θ,T_k} where $T_k = \inf\{t : Y_t < 2^{-k}\}$. But then since T is the a.s. limit of the increasing sequence $\langle T_k\rangle_{k\in\mathbb{N}}$, $K^{\theta,T}$ is a semimartingale (see [17]). Observe that $M = A$ on $\{Y_- = 0\}$ and therefore

[3] This claim is now recognized as being incorrect.

$\int\{Y_- = 0\}dM$ is a predictable local martingale of integrable variation and therefore null. The process Z in (11) is thus still well defined and by (12) it vanishes when Y does. Again by integration by parts we conclude that $ZK^{\theta,T} = ZK^{\theta}$ is a local martingale.

It should not be too surprising that under (2), the martingale density may fail to be strictly positive. The corresponding situation in the context of martingale *measures* was illustrated in a well known example (see [6, Example 7.7]), then further considered for the case of continuous price process in [7]. In [7], it was actually shown that the absence of arbitrage opportunities implies the existence of a martingale density. Moreover, this was associated to an absolutely continuous local martingale measure. However, both implications require arbitrage opportunities to be defined with respect to general integrands, not just simple processes. Corollary 1 is therefore of interest on its own.

In order to further clarify the relationship between arbitrage opportunities, free lunches and martingale densities we conclude with the following result, already mentioned in [7], but of which we offer a new proof.

Lemma 3. *If there are no arbitrage opportunities and there exists a strictly positive martingale density, then there is no free lunches.*

Proof. We recall the following fact [6, Proposition 3.5]: Under (2), $\|K^{\theta}_{\infty} \wedge 0\| \geq \|K^{\theta}_{\tau} \wedge 0\|$ for each $\tau \in \mathcal{T}$. Let $\langle \tau_n \rangle_{n\in\mathbb{N}}$ be a localizing sequence of stopping times along which Z is a uniformly integrable martingale. Then

$$0 = \lim_n P(Z_{\tau_n} K^{\theta}_{\tau_n}) = \lim_n P\big(Z_{\tau_n}(K^{\theta}_{\tau_n} + \|K^{\theta}_{\infty} \wedge 0\|)\big) - \|K^{\theta}_{\infty} \wedge 0\| \geq P(Z_{\infty}K^{\theta}_{\infty}) - \|K^{\theta}_{\infty} \wedge 0\|$$

i.e. $P(Z_{\infty}K^{\theta}_{\infty}) \leq \|K^{\theta}_{\infty} \wedge 0\|$. If $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$ (so that $K^{\theta_n}_{\infty} \geq x_n$ for some $\theta_n \in \Theta$) converging in L^{∞} to $x \geq 0$, then

$$P(Z_{\infty}x) = \lim_n P(Z_{\infty}x_n) \leq \lim_n P(Z_{\infty}K^{\theta_n}_{\infty}) \leq \lim_n \|K^{\theta_n}_{\infty} \wedge 0\| \leq \lim_n \|x_n \wedge 0\| \leq \lim_n \|x - x_n\|$$

But then if $P(Z_{\infty} = 0) = 0$ one is forced to conclude that $P(x = 0) = 1$.

4 Comments

A martingale density, except if of class D, does not induce a martingale measure, a basic tool in asset pricing. Further, even when the martingale density is strictly positive, it is in general still not sufficient to exclude arbitrage opportunities. There are several versions of the $FTAP$ which ensure the existence of martingale measure. However, even though these versions possess more elegant mathematical properties, they are more restrictive than that considered here. To our knowledge, there has been no previous rigorous characterization of martingale densities on the basis of the no arbitrage principle. The existence of a martingale measure may be obtained either by enlarging the set of admissible trading strategies, as in [6] , or by adopting a definition of free lunch more restrictive than the one adopted above, as in [1] and [22].

It should be remarked, however, that despite its desirable implications, the existence of a martingale measure places considerable constraints on the price process, particularly on volatility. An example of this is provided by the so called *strong non degeneracy* condition imposed, among others, in [5, p. 654], and consisting in a lower bound on asset volatility that guarantees the martingale nature of the market price of risk – or even its square integrability. However when we come to financial modelling, volatility is a key element in the explanation of some of the stylized facts of finance. As a consequence, although all financial models invariably admit a martingale density, those which admit a martingale measure are hardly the case.

Further restrictions to asset pricing models are implicit in the existence of a martingale measure Q. The pricing formula $S_0 = Q(S_{\infty})$ applicable to all models in which the discounted price process is bounded, precludes the existence of pricing bubbles. However, when prices are positive and Z is only a martingale density, a straightforward application of the Fatou lemma delivers $S_0 \geq P(Z_\infty S_\infty)$. As remarked in [15], it is reasonable to interpret $P(Z_\infty S_\infty)$ as the fundamental value of the asset and, consequently, the quantity $\beta(S) = S_0 - P(Z_\infty S_\infty)$ as the "bubble" part of the asset price. In [15], however, it is assumed that the martingale density is strictly positive but it is far from clear how this relates to the no arbitrage principle.

5 Conclusions

After proving that the Yan theorem remains valid after replacing L^p, $1 \leq p < \infty$, by L^∞, we have considered a financial market characterized by a continuous vector process S describing asset prices in discounted units and in the absence of free lunches. These have been defined in terms of the L^∞ topology in a way similar to [6]. In Proposition 1 we have proved that under these assumptions asset prices are necessarily semimartingales and there exists a strictly positive martingale density, i.e. a local martingale Z such that $Z_0 = 1$, $P(Z_\infty > 0) = 1$ and that ZS is a local martingale. The novelty of this result is that it is formulated in terms of the density process Z rather than of a martingale measure which, under our assumptions, need not exist. Although the existence of a martingale measure is indeed a desirable property, it is typically obtained by imposing stringent constraints on the volatility process and in fact most models fail to satisfy it. Reformulating our problem in terms of arbitrage opportunities rather than free lunches, we are able to arrive at the weaker conclusion that there exists a martingale density Z and that prices, stopped by the time Z expires, are semimartingales. Even when Z is strictly positive, the existence of a martingale density is not sufficient to exclude arbitrage opportunities.

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